MATH 140A Review: Sequences and Series

1. Determine if the following series converges. If so, what is the sum?

$$\sum_{n=2}^{\infty} \left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}} \right).$$

Solution: Let

$$\begin{aligned} S_N &= \sum_{n=2}^{N} \left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}} \right) \\ &= \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} \right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} \right) + \cdots + \left(\frac{1}{\sqrt{N-1}} - \frac{1}{\sqrt{N}} \right) \\ &= \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{N}}. \end{aligned}$$

Hence,

$$\sum_{n=2}^{\infty} \left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}} \right) = \lim_{N \to \infty} S_N = \lim_{N \to \infty} \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{N}} \right) = 1.$$

Hence, the series converges to 1.

2. Determine if the following series converges. If so, what is the sum?

$$\sum_{n=1}^{\infty} \left(\pi^{-n} + 2^{n+1} 4^{-n} \right).$$

Solution: We have that

$$\sum_{n=1}^{\infty} \pi^{-n} = \sum_{n=0}^{\infty} \pi^{-n-1} = \frac{1}{\pi} \sum_{n=0}^{\infty} \left(\frac{1}{\pi} \right)^n = \frac{1}{\pi(1 - 1/\pi)},$$

where the last holds by geometric series formula for $r = 1/\pi$. We also have that

$$\sum_{n=1}^{\infty} 2^{n+1} 4^{-n} = \sum_{n=1}^{\infty} 2 \left(\frac{2}{4} \right)^n = 2 \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^{n+1} = \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n = \frac{1}{1 - 1/2}$$

where the last holds by geometric series formula for $r = 1/2$. Hence,

$$\sum_{n=1}^{\infty} \left(\pi^{-n} + 2^{n+1} 4^{-n} \right) = \frac{1}{\pi - 1} + 2.$$

Thus, the series converges.

3. If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$S_N = 2 + e^{-N},$$

then find a_n and $\sum_{n=1}^{\infty} a_n$.

Solution: Note that $S_{N+1} = S_N + a_{N+1}$. Hence, $2 + e^{-N-1} = 2 + e^{-N} + a_{N+1}$. Therefore, $a_{N+1} = e^{-N-1} - e^{-N}$. That is, $a_n = e^{-n} - e^{-n-1}$. We then have that

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} (2 + e^{-N}) = 2.$$

Thus, the series converges to 2.